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Aurifeuillian factorizations and the period of the Bell numbers modulo a prime

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ABSTRACT. We show that the minimum period modulo p of the Bell exponential integers is $(p^p - 1)/(p-1)$ for all primes $p < 102$ and several larger p. Our proof of this result requires the prime factorization of these periods. For some primes p the factoring is aided by an algebraic formula called an Aurifeuillian factorization. We explain how the coefficients of the factors in these formulas may be computed.

1. Introduction

The first order Bell exponential integers $B(n)$ may be defined by the generating function

$$e^{e^x-1} = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}.$$

These integers appear in many combinatorial problems. For example, $B(n)$ is the number of ways a product of n different primes may be factored. See [**6**] and its references for more background.

Williams [**13**] proved that for each prime p the sequence $\{B(n) \bmod p; n = 0, 1, \dots\}$ is periodic and that the minimum period divides

$$N_p = \frac{p^p - 1}{p - 1}.$$

He showed that the minimum period is precisely N_p for $p = 2$, 3 and 5. Levine and Dalton [**6**] showed that the minimum period is exactly N_p for $p = 7$, 11, 13 and 17. They also investigated the period for the other primes < 50. We show that the minimum period is exactly N_p for each prime < 102 and for several larger primes. Our technique is the same one used by Levine and Dalton. We show that the period is not N_p/q for any prime factor q of N_p. We were able to extend their work so far because of great advances in integer factoring methods since 1962.

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In the next two sections we describe our attempts to factor N_p for primes $p < 180$. The final section explains how we investigated the period of $\{B(n) \bmod p\}$.

2. Factorization of N_p

As we tried to factor N_p for the odd primes $p < 180$, we also tried to factor the important related numbers $K_p = (p^p + 1)/(p + 1)$ for the same primes p. It is well known that all prime factors of N_p and K_p have the form $2kp + 1$, where k is a positive integer. After just a little trial division we used the Elliptic Curve Method [**5**]. We used the Quadratic Sieve Method [**9**] to factor the occasional integer of modest size which did not succumb to the Elliptic Curve Method. Before we did any of this work, however, we used the fact that for each odd prime p, one of N_p, K_p admits an algebraic factorization into two nearly equal factors. In fact, if p is squarefree, then the numbers $(p^{hp} - 1)/(p - 1)$ when $p \equiv 1 \pmod 4$ and $(p^{hp} + 1)/(p + 1)$ when $p \equiv 2$ or $3 \pmod 4$ have algebraic factorizations for all odd h. Although we describe these factorizations in general in Theorem 2, in this paper we use only the case $h = 1$ and p prime. The algebraic factorizations are called *Aurifeuillian* because some of these formulas were discovered by Aurifeuille (see page 276 of [**7**]).

The known factors of N_p and K_p are given in Tables 1 and 2. The notations Pxx and Cxx denote prime and composite numbers of xx digits. An L or M following p refers to the Aurifeuillian factor of Theorem 2 below.

Levine and Dalton [**6**] copied some factors from the table in Cunningham [**4**] including the erroneous "factor" 6709 of N_{43}, and found more factors by trial division. But they did not use the Aurifeuillian factorizations from [**4**]. If they had, they could have finished factoring N_{29} and probably also N_{37}.

3. Aurifeuillian factorizations

For integers $n > 0$ let $\Phi_n(x)$ denote the cyclotomic polynomial

$$\Phi_n(x) = \prod_{\substack{j=1 \\ (j,n)=1}}^{n} (x - \zeta_n^j),$$

where ζ_n is a primitive n-th root of unity. It is well known that $x^n - 1 = \prod_{d|n} \Phi_d(x)$. If p is an odd prime, then $(x^p - 1)/(x - 1) = \Phi_p(x)$ and $(x^p + 1)/(x + 1) = \Phi_{2p}(x)$. Thus $N_p = \Phi_p(p)$ and $K_p = \Phi_{2p}(p)$. Although $\Phi_n(x)$ is irreducible over the integers, it may be reducible over certain quadratic fields. Theorem 1 sets the stage for some factorizations of this type. The first two parts of Theorem 1 were proved by Lucas [**8**]. Schinzel [**11**] gave a modern proof of the entire theorem. Our Theorem 1 is the case $m = n$ of Theorem 1 of [**11**]. Let $(m|n)$ be the Jacobi symbol. For $\sqrt{c}$ we make the convention $\sqrt{c} \geq 0$ if $c \geq 0$ and $\sqrt{c} = i\sqrt{-c}$ if $c < 0$.

THEOREM 1. *Let $n > 1$ be a squarefree integer. Then there exist polynomials $P_n(x)$ and $Q_n(x)$ with integer coefficients such that*

$$\Phi_n(x) = P_n^2(x) - (-1|n)nxQ_n^2(x) \quad \textit{and} \quad \Phi_{2n}(x) = P_n^2(-x) + (-1|n)nxQ_n^2(-x)$$

Table 1: Factorization of $N_p = (p^p - 1)/(p - 1)$ for primes $p < 180$.

p	Known prime factors of N_p
3	13
$5L$	11
$5M$	71
7	29.4733
11	15797.1806113
$13L$	1803647
$13M$	53.264031
$17L$	2699538733
$17M$	10949.1749233
19	1099122030922396438 40221
23	461.1289.831603031789.1920647391913
$29L$	84449.2428577.549334763
$29M$	59.16763.14111459.58320973
31	568972471024107865287021434301977158534824481
$37L$	149.4190342555354483999815 8239
$37M$	1999.7993.16651.17317.1019271565 6759
$41L$	1752341.20567159.18768593110908 03007
$41M$	83.5926187589691497537793497756719
43	173.120401.$P62$
47	1693.255742492896763511474638530188876017.$P39$
$53L$	107.16505521259654533.143470720478589313288313473
$53M$	141829.130339605796313248804554498814089943 92143
59	709.141579233.$P92$
$61L$	977.343625872243632312073.398853286456071792609917995907
$61M$	1000403244183535565720394723140528028235711874491322863
67	269.4021.730837.10960933.
	.151495488509660402356228791573 0049.$P69$
71	105649.3388409395214741.17882954877203881.$P93$
$73L$	1414741.1295720382587.1192167517020392933.$P31$
$73M$	293.439.25239167.56377463.3611379501352361.$P32$
79	317.1558537597.171355071830508389477.
	.544931329080433782632029 13.$P91$
83	2657.11155201.100850570760132334915676 9489.$P120$
$89L$	179.800986210355770 9.596484421043200640 7836201.$P43$
$89M$	37307598912253490893302199133.$P58$
$97L$	$P95$
$97M$	389.363751.684640163.119437287337412947643906 02153.$P51$

Table 1 continued.

p	Known prime factors of N_p
$101L$	$1213.9931988588681.102208068907493.393101595766008847.P53$
$101M$	$607.5657.157561.P89$
103	$1237.16706917226363953216841.C180$
107	$137122213.10508824813.C197$
$109L$	$2617.C107$
$109M$	$6196098743139082891438631.P86$
$113L$	$3391.8363.78519280025619789864443171 4786031.P75$
$113M$	$227.34314816732569.$
	$.7073925576907761667406608531803081165593 2920203.P53$
127	$509.22861.1320675600886906675359917.C234$
131	$1049.1742643541410742623061.C251$
$137L$	$5414288355738338318013979 1.C120$
$137M$	$1097.124123.1918644449.12779722229.574894288613.$
	$.271329112787027.17594294674609358799167756 10180659.P59$
139	$557.119833345601.C282$
$149L$	$1193.C158$
$149M$	$51784951.450090559.465814231.C137$
151	$2417.15101.1234577.C314$
$157L$	$1356984109417.C159$
$157M$	$86351.P167$
163	$653.2609.41729.31943437.3727539197017.391683908074297.$
	$.8224734227858383253.P294$
167	$16033.1001953110409.66980625067862951404 5626189.P326$
$173L$	$347.685081.P184$
$173M$	$161297590410850151.P176$
179	$C402$

Table 2: Factorization of $K_p = (p^p + 1)/(p + 1)$ for primes $p < 180$.

p	Known prime factors of K_p
3	7
5	521
$7L$	113
$7M$	911
$11L$	58367
$11M$	23.89.199
13	13417.20333.79301
17	45957792327018709121
$19L$	108301.1049219
$19M$	870542161121
$23L$	47.139.1013.52626071
$23M$	1641281.1522029233
29	233.6864997.9487923853.563966387871654508723 3
$31L$	1613.145577.35789156484227
$31M$	373.62869.270669020246864 9
37	593.134135213.4356032201.6190006021.$P27$
41	18041.20396681.$P53$
$43L$	947.6709.114083480416893545462206737 7
$43M$	1291.86689.48592600897222666433103668 3
$47L$	65519.10519189757.60963223421.2506611914519
$47M$	659.15511.21179047.354341392424904982208989 3
53	991313.2644277.5324593.14443842647093.19604216783737.$P45$
$59L$	27759619.6806872605199.439371719230866406886584144374 1
$59M$	4466419.11821911653180627.11488862755597074594499643626 3
61	2441.1191941.9229762307875553.560622532089629629. .2852371693967589142 7869.$P42$
$67L$	$P60$
$67M$	141907.4002983.5759607944561.$P37$
$71L$	4872163.72704953628310243647543552 87.$P30$
$71M$	17467.59743093.$P54$
73	459636916558529111235282963785233915709014470880783267 7.$P80$
$79L$	$P74$
$79M$	34919.188021.45780868646549.$P51$
$83L$	499.9463.$P72$
$83M$	167.997.17929.472168956426245957.86078539587433148743 1.$P32$
89	169573582127857.111884572111315134368315395 01.$P130$
97	1553.16318716076815740 53.$C170$

Table 2 continued.

p	Known prime factors of K_p
101	10741549365517.266345719946724536329.C167
103L	2267.18541.237313.435777501586491 83. .11332178618362834297825839691308092 53.P37
103M	1031.692779.367338623155396247970229930148464620 17.P57
107L	1061227.46242619.304535269. .32116109518801441836697852196938078 57.P49
107M	643.2121939803795871061.228662026524021137787 7.P66
109	1236165024989.103413887493374456170 33.P186
113	2713.108637220969.761996288465571689 21.C196
127L	921259.152523854179855862280920221 3.P99
127M	5932933.267590103252555719351094 71.P101
131L	263.8123.23581.128119.509192023.5434194401.118531075451349793. .22748277370249933900204468376 27.P56
131M	3407.16003103839.842581814842187453048134381 7. .40597046694975803542870745682 1.P67
137	136453.164095915779277.C272
139L	25577.C144
139M	21374190911672122661.197718513453774939657 7.P108
149	3513009953.490746610814080698 1.915115125488764974144697. .28094398708254247143685653 13.C242
151L	53593223.20110202953.322631539451020618739. .21410447638232281941934857667.P97
151M	7853.C160
157	P343
163L	P179
163M	6521.185821.2272547.21163569551.C154
167L	3760684691.14974117420259.C162
167M	8017.3295913.465247639.4386303138831827.C151
173	C385
179L	359.1433.909679.11306999215101373913622 7.P166
179M	1597039.586442063977132703776 9.C173

when n is odd and

$$\Phi_{2n}(x) = P_n^2(x) - nxQ_n^2(x)$$

when n is even. These polynomials can be computed from the formulas

$$P_n(x^2) - \sqrt{(-1|n)n}\, xQ_n(x^2) = \prod_s (x - \zeta_n^s) \prod_t (x + \zeta_n^t)$$

$$P_n(-x^2) - i\sqrt{(-1|n)n}\, xQ_n(-x^2) = \prod_s (x + i\zeta_n^s) \prod_t (x - i\zeta_n^t)$$

where the products are over $0 < s < n$, $0 < t < n$, $(st, n) = 1$, $(s|n) = 1$, $(t|n) = -1$ when n is odd and from the formula

$$P_n(x^2) - \sqrt{n}\, xQ_n(x^2) = \prod_s (x - \zeta_{4n}^s)$$

where the product is over $0 < s < 4n$, $(s, 4n) = 1$, $(n|s) = 1$ when n is even.

It is easy to modify Theorem 1 to use only real numbers. Theorem 2 does this and also restricts the identities to cases when they produce interesting Aurifeuillian factorizations, that is, when the cyclotomic polynomial is expressed as the difference of two squares. Let $\phi(n)$ denote Euler's totient function.

THEOREM 2. *Let $n > 1$ be an odd squarefree integer. Then there exist polynomials $C_n(x)$ and $D_n(x)$ with integer coefficients and degrees $\phi(n)/2$ and $\phi(n)/2 - 1$, respectively, with the following properties. Let h be an odd positive integer. If $n \equiv 1 \pmod 4$, then*

$$\Phi_n(n^h) = (C_n(n^h) - n^{(h+1)/2} D_n(n^h))(C_n(n^h) + n^{(h+1)/2} D_n(n^h)),$$

and if $n \equiv 3 \pmod 4$, then

$$\Phi_{2n}(n^h) = (C_n(n^h) - n^{(h+1)/2} D_n(n^h))(C_n(n^h) + n^{(h+1)/2} D_n(n^h)). \tag{1}$$

The coefficients of $C_n(x)$ and $D_n(x)$ may be computed from the identity

$$C_n(x^2) - \sqrt{n}\, xD_n(x^2) = \prod_{\substack{s=1 \\ (s,n)=1}}^{(n-1)/2} (x^2 - 2(s|n) f_n(s) x + 1) \tag{2}$$

where $f_n(s) = \cos \frac{2\pi s}{n}$ if $n \equiv 1 \pmod 4$ and $f_n(s) = \sin \frac{2\pi s}{n}$ if $n \equiv 3 \pmod 4$.

Let n be an even squarefree positive integer. Then there exist polynomials $C_n(x)$ and $D_n(x)$ with integer coefficients and degrees $\phi(n)$ and $\phi(n) - 1$, respectively, so that (1) holds when h is an odd positive integer. The coefficients of $C_n(x)$ and $D_n(x)$ may be computed from the identity

$$C_n(x^2) - \sqrt{n}\, xD_n(x^2) = \prod_{\substack{s=1 \\ (s,n)=1}}^{2n} (x^2 - (1 + (n|s)) \cos \frac{\pi s}{2n} x + 1).$$

PROOF. Let $n \equiv 1 \pmod 4$. Then $(-1|n) = 1$. By Theorem 1, $\Phi_n(x) = P_n^2(x) - nxQ_n^2(x)$ where

$$P_n(x^2) - \sqrt{n}\,xQ_n(x^2) = \prod_{\substack{s=1\\(s,n)=1}}^{n-1} (x - (s|n)\zeta_n^s).$$

In the product combine the factors with s and $n-s$. Note that $(s,n) = 1$ if and only if $(n-s,n) = 1$. Also $(n-s|n) = (s|n)$ and $\zeta_n^s + \zeta_n^{n-s} = 2\cos\frac{2\pi s}{n}$. The product of the two factors is $x^2 - 2(s|n)2\cos\frac{2\pi s}{n}x + 1$. Writing $C_n(x) = P_n(x)$, $D_n(x) = Q_n(x)$ and $x = n^h$ gives the result. There are $\phi(n)/2$ quadratic factors in the product in (2) so the degree of the polynomial in (2) is $\phi(n)$. Since this polynomial is $C_n(x^2) - \sqrt{n}\,xD_n(x^2)$ the degree of C_n is $\phi(n)/2$ and the degree of D_n is $\phi(n)/2 - 1$.

Now let $n \equiv 3 \pmod 4$. Then $(-1|n) = -1$. By Theorem 1, $\Phi_{2n}(x) = P_n^2(-x) - nxQ_n^2(-x)$ where

$$P_n(-x^2) - i\sqrt{n}\,xQ_n(-x^2) = \prod_{\substack{s=1\\(s,n)=1}}^{n-1} (x + i(s|n)\zeta_n^s).$$

In the product combine the factors with s and $n-s$. Note that $(s,n) = 1$ if and only if $(n-s,n) = 1$. Also $(n-s|n) = -(s|n)$ and $\zeta_n^s - \zeta_n^{n-s} = 2i\sin\frac{2\pi s}{n}$. The product of the two factors is $x^2 - 2(s|n)2\sin\frac{2\pi s}{n}x + 1$. Writing $C_n(x) = P_n(-x)$, $D_n(x) = Q_n(-x)$ and $x = n^h$ gives the result.

Now suppose n is even. Then $n \equiv 2 \pmod 4$ because n is squarefree. By Theorem 1, $\Phi_{2n}(x) = P_n^2(x) - nxQ_n^2(x)$ where

$$P_n(x^2) - \sqrt{n}\,xQ_n(x^2) = \prod_{\substack{s=1\\(s,4n)=1\\(n|s)=1}}^{4n} (x - \zeta_{4n}^s)$$

In the product combine the factors with s and $4n-s$. Note that $(s,4n) = 1$ if and only if $(4n-s,4n) = 1$. Also $(n|4n-s) = (n|s)$ and $\zeta_{4n}^s + \zeta_{4n}^{4n-s} = 2\cos\frac{2\pi s}{4n}$. The product of the two factors is $x^2 - (1+(n|s))\cos\frac{\pi s}{2n}x + 1$. Since $(n,s) = 1$ the factor $(1+(n|s))$ is 2 when $(n|s) = 1$ and is 0 when $(n|s) = -1$. Writing $C_n(x) = P_n(x)$, $D_n(x) = Q_n(x)$ and $x = n^h$ gives the result and proves Theorem 2.

The two factors of $\Phi_n(n)$ or $\Phi_{2n}(n)$ in Theorem 2 are denoted nL and nM in Tables 1 and 2. A table of coefficients of $C_n(x)$ and $D_n(x)$ for $n < 120$ may be found in Table 34 on page 453 ff. of Riesel [**10**].

Ordinary 64-bit double precision floating point arithmetic permits the correct calculation in a fraction of a second of these coefficients for odd $n < 180$. The program was tested by comparing the product of nL and nM, computed from $C_n(n)$ and $D_n(n)$, with N_n or K_n, computed independently.

Brent [**2**] gives an algorithm for computing the coefficients of $C_n(x)$ and $D_n(x)$ which uses integer arithmetic throughout.

4. The period of $\{B(n) \bmod p\}$

When p is prime, this period is known to be a divisor of N_p. To test whether the period divides some factor N of N_p, it is enough to compare $B(N+i) \bmod p$ with $B(i) \bmod p$ for $1 \le i \le p$. Only these p pairs need to be compared because the congruence

$$B(n+p) \equiv B(n) + B(n+1) \pmod{p} \tag{3}$$

of Touchard [**12**] shows that any p consecutive values of $B(n) \bmod p$ determine the sequence after that point. For each prime divisor q of N_p listed in Table 1, the test just described was performed for $N = N_p/q$. When we could not factor N_p completely, we performed the test also for N_p divided by its remaining composite cofactor q (two of them for N_{149}). In every case the outcome of the test was that the period did not divide N_p/q. We also performed the test with $N = N_p$ to check the program. The not unexpected outcome was that N_p is a period. Finally, we tested some N_p with $p > 180$ to see whether the period might be slightly smaller than N_p. Specifically, for each prime p in $180 < p < 1100$ we computed all primes $q < 2^{31}$ dividing N_p and tested N_p/q for being a period. It never was a period. Thus, we have proved the following result.

THEOREM 3. *The minimum period of the sequence $\{B(n) \bmod p\}$ is N_p when p is a prime < 102 and also when $p = 113$, 163, 167 or 173. For the remaining primes $p < 180$, no proper divisor of N_p whose codivisor appears in Table 1 is a period of the sequence. Furthermore, for each prime $p < 1100$, no proper divisor of N_p whose codivisor has only prime factors $< 2^{31}$ is a period of the sequence.*

Based on the evidence provided by Theorem 3, we conjecture that the minimum period of the sequence $\{B(n) \bmod p\}$ is N_p for every prime p.

It remains to explain how we computed $B(N) \bmod p$ when p is a prime < 1100 and N is large; some N have thousands of decimal digits. First of all, we computed $b_i = B(i) \bmod p$ for $0 \le i \le p$ using the formula $B(n+1) = \sum_{j=0}^{n} \binom{n}{j} B(j)$ of Cesàro [**3**] (see also Becker and Browne [**1**]). That is, we used this algorithm:

```
b_0 = 1;
b_1 = 1;
t_0 = 1;
for j = 2 to p do
    begin
    t_{j-1} = b_{j-1};
    for i = j - 2 down to 0
        t_i = (t_i + t_{i+1}) mod p;
    b_j = t_0;
    end
```

This algorithm takes $O(N^2)$ operations to compute $B(N) \bmod p$, so it is too slow to use for large N. To compute $B(N) \bmod p$ for large N we use the congruence $B(n+p^m) \equiv B(n+1)+mB(n) \pmod{p}$ of Touchard [**12**], which generalizes (3). We write N in radix p as $N = \sum_{i=0}^{e} a_i p^i$, where $0 \le a_i < p$ and $a_e \ne 0$. Starting from the block $b_i = B(i) \bmod p$, $0 \le i \le p$ we use the digits a_i to compute other blocks of length $p+1$ of values of $B(i) \bmod p$. The algorithm, which runs in $O(p^2 \log N)$ steps, is:

```
for i = 0 to p
    c_i = b_i;
for i = 1 to e
    if a_i > 0 then
        begin
        for j = 1 to a_i do
            begin
            for k = 0 to p - 1
                d_k = (c_{k+1} + i * c_k) mod p
            d_p = (d_0 + d_1) mod p
            for k = 0 to p
                c_k = d_k
            end
        end
```

At this point c_{a_0} is $B(N) \bmod p$. Use (3) to shift the window to $B(N+i) \bmod p$ for $0 \le i \le p$. Then compare $B(N+i) \bmod p$ with b_i for $0 \le i \le p$ to decide whether N is a period. For every proper divisor N of N_p that we examined, either $B(N) \bmod p \ne b_0$ or $B(N+1) \bmod p \ne b_1$.

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